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Abstract

We discuss some aspects of a new noncombinatorial fermionic approach to the two-dimensional dimer problem in statistical mechanics based on the integration over anticommuting Grassmann variables and factorization ideas for dimer density matrix. The dimer partition function can be expressed as a Gaussian fermionic integral. For regular lattices, the analytic solution then follows by passing to the momentum space for fermions.

1 Introduction

The dimer problem was originally formulated to describe the entropy of diatomic molecules adsorbed on a crystal surface [1]. In a more abstract sense, the two-dimensional (2D) dimer model is one of only few nontrivial exactly solvable models in statistical mechanics [2-8]. In its original state, this is a purely combinatorial problem, close in its inherent spirit to the 2D Ising model [4]. The exact solution for the closed-packed 2D dimer problem on the standard rectangular lattice was first obtained by a direct combinatorial method [2,3]. A number of more complicated dimer lattices have also been analyzed [5,6]. Traditional combinatorial approaches of this kind, however, are somewhat complicated, and are different, in their spirit, from the methods commonly used in condensed matter physics. In this report we

comment on a simple fermionic approach to the 2D dimer problem based on the integration over anticommuting Grassmann variables (nonquantum fermionic fields) and the mirror-ordered factorization procedure for the dimer density matrix [7,8]. Neither transfer-matrices nor combinatorics are used. The factorization method rather resembles the idea of insertion of the Dirac unity, $\sum |a\rangle\langle a| = 1$, when making transformations in quantum mechanics. We first reformulate the combinatorial dimer partition function, Q , in terms of the commuting nil-potent (η)-variables, and then introduce into the resulting expression the anticommuting (a)-variables by factorization of the local bond Boltzmann weights. Eliminating the η -variables in the resulting (η, a)-representation, we come to a purely fermionic expression for the same partition function Q . Even for the inhomogeneous distribution of the dimer weights over the lattice, the partition function is expressible as a Gaussian fermionic integral. Equivalently, the closed-packed 2D dimer problem is a theory of free fermions on a lattice. For regular lattices, the analytic solution then follows by transformation to the momentum space for fermions.

2 Grassmann variables

Let us remember first the basic rules of anticommuting analysis [9]. Grassmann variables are classic (nonquantum) fermionic numbers purely anticommuting to zero. Given a set of Grassmann variables a_1, \dots, a_N , we have: $a_i a_j + a_j a_i = 0$, $a_j^2 = 0$. The first important identity is that for the product of linearly transformed variables:

$$b_1 b_2 \dots b_N = \det A \cdot a_1 a_2 \dots a_N, \quad b_i = \sum_{j=1}^N A_{ij} a_j, \quad (2.1)$$

where A is the matrix of transformation $a \rightarrow b$. The determinant here appears due to the known interrelations between fermionic algebra and determinant combinatorics. The rules of fermionic integration were first formulated by Berezin [9]. For one variable, we have:

$$\int da_j \cdot a_j = 1, \quad \int da_j \cdot 1 = 0. \quad (2.2)$$

In the multiple fermionic integral the differential symbols da_1, \dots, da_N are again anticommuting with each other and with the variables [9]. Due to the property $a_j^2 = 0$, any natural function defined on the set a_1, \dots, a_N can be written as a finite polynomial in these variables:

$$f(a_1, a_2, \dots, a_N) = f_0 + \sum_{(1 \leq j \leq N)} f_j a_j + \dots + f_{123\dots N} a_1 a_2 \dots a_N, \quad (2.3)$$

where $f_0, \dots, f_{123\dots N}$ are numerical parameters. Integrating a polynomial function like (2.3) according to (2.2), we obtain:

$$\int da_N \dots da_2 da_1 f(a_1, a_2, \dots, a_N) = f_{123\dots N}, \quad (2.4)$$

where $f_{123\dots N}$ is just the coefficient in the last term of the polynomial (2.3). We see that fermionic integration is a trivial problem if the integrand function is already known in a polynomial form. However, this may be not the case in applications. From (2.1) and (2.4), one can deduce the rules of change of variables in a fermionic integral under a linear substitution. As compared with the rules of a proper commuting analysis the only difference is that the Jacobian will now appear in the inverse power.

An important role in applications play Gaussian fermionic integrals [9,10]. The Gaussian integral of the first kind is related to the determinant:

$$\int \prod_{j=1}^N da_j^* da_j \exp \left(\sum_{i=1}^N \sum_{j=1}^N a_i A_{ij} a_j^* \right) = \det \hat{A}, \quad (2.5)$$

where $\{a_j, a_j^*\}$ is a set of purely anticommuting Grassmann variables, the matrix \hat{A} is arbitrary. By convention, the variables a_j and a_j^* can be considered as complex-conjugated fermionic fields, otherwise these are independent variables. The fermionic exponential is assumed in the sense of its series expansion, the series terminates at some stage due to the property $a_j^2 = 0$. The correspondent finite polynomial also can be obtained by multiplying elementary factors $\exp(a_i A_{ij} a_j^*) = 1 + a_i A_{ij} a_j^*$. The Gaussian integral of the second kind, for real fermionic fields, is related to the Pfaffian:

$$\int da_N \dots da_2 da_1 \exp \left(\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_i A_{ij} a_j \right) = \text{Pfaff } \hat{A}, \quad A + A^T = 0, \quad (2.6)$$

where the matrix \hat{A} is now skew-symmetric ($A_{ij} + A_{ji} = 0$, $A_{ii} = 0$). The Pfaffian is some combinatorial polynomial in elements A_{ij} well known in mathematics for a long time. In fact, the Pfaffian combinatorics is identical with that of fermionic Wick's theorem. The Pfaffian and determinant of the associated skew-symmetric matrix are algebraically related: $\det \hat{A} = (\text{Pfaff } \hat{A})^2$. This identity can be most readily proved just in terms of integrals like (2.5) and (2.6), making use of a suitable change of variables in (2.5).

3 The dimer problem

Let us consider the closed-packed 2D dimer model on a rectangular lattice net with the most general inhomogeneous distribution of dimer weights [7,8]. The dimers are objects living on lattice bonds. A given bond may be either free or covered by a dimer together with the two adjacent lattice sites. The closed-packing condition means that each lattice site must be occupied by one and only one dimer. The lattice must have an even number of sites to be covered completely by dimers in a closed-packed fashion. An example of a closed-packed dimer configuration on a rectangular net can be seen in Fig. 1a in [7].

Let the lattice sites be numbered by the integer Cartesian coordinates mn , where $m = 1, \dots, M$, $n = 1, \dots, N$ run in horizontal and vertical directions, respectively. MN is the total number of sites. We define $t_{mn}^{(1)}$ and $t_{mn}^{(2)}$ to be the dimer weights (activities) for horizontal ($mn | m+1n$) and vertical ($mn | mn+1$) bonds, respectively. The weight of a free bond is 1. The dimer partition function is:

$$Q = \sum_{\{N\}} \left\{ \prod_{m=1}^M \prod_{n=1}^N t_{mn}^{(1) N_{mn}^{(1)}} t_{mn}^{(2) N_{mn}^{(2)}} \right\}, \quad (3.1)$$

where $N_{mn}^{(1)}$ and $N_{mn}^{(2)}$ are dimer occupation numbers taking the values 1, 0, depending of whether a given bond is covered or free, and the sum is taken over all closed-packed configurations. Notice that in a completely homogeneous and symmetric case, $t_{mn}^{(1)} = t_{mn}^{(2)} = 1$, the partition function yields simply the total number of closed-packed dimer configurations for a given lattice. The thermodynamic interpretation is only possible if there are few sorts of dimers with different weights (energies).

We need to transform Q into a fermionic Gaussian integral. The combinatorial definition (3.1) is not very suitable in this respect since the symbol of the sum in (3.1) is only a conventional prescription for counting over closed-packed configurations. In fact, we need some kind of a formal averaging which will select the closed-packed configurations automatically. This can be realized in terms of the commuting nilpotent variables [7]. With each lattice site we now associate the commuting nilpotent variable η_{mn} , such that $\eta_{mn}^2 = 0$, and write:

$$Q = Sp_{(\eta)} \left\{ \prod_{m=1}^M \prod_{n=1}^N (1 + t_{mn}^{(1)} \eta_{mn} \eta_{m+1n}) (1 + t_{mn}^{(2)} \eta_{mn} \eta_{mn+1}) \right\}, \quad (3.2)$$

with the averaging rules for one variable:

$$Sp_{(\eta_{mn})} (1 | \eta_{mn} | \eta_{mn}^2 | \eta_{mn}^3 | \dots) = (0 | 1 | 0 | 0 | \dots). \quad (3.3)$$

The global η -averaging in (3.2) assumes local averagings like (3.3) at all sites. In (3.2) we also assume the free-boundary conditions: $\eta_{M+1n} = \eta_{mN+1} = 0$. Expression (3.2) exactly reproduces the combinatorial partition function (3.1). The product of the Boltzmann factors $1 + t\eta\eta'$ forming the density matrix in (3.2) generates all possible dimer coverings of a lattice by ‘dimer molecules’, $\eta t \eta'$, with no restrictions, while the η -averaging according to (3.3) selects the closed-packed configurations. These are the configurations in which each lattice site is covered by one and only one ‘atom’ or ‘monomer’, η_{mn} , which can only be a part of some dimer.

4 Fermionization

Let it be given two Grassmann variables, a and a^* . The elementary Gaussian exponential is $e^{\lambda aa^*} = 1 + \lambda aa^*$, where λ is a parameter. Noting (2.2), for the associated Gaussian averages we then find: $\int da^* da e^{\lambda aa^*} \{1, a, a^*, aa^*\} = \{\lambda, 0, 0, 1\}$. Making use of these rules, we can factorize the weight factors from (3.2) as is shown in (4.1) below. For the whole lattice, we introduce a set of purely anticommuting Grassmann variables, $\{a_{mn}, a_{mn}^*, b_{mn}, b_{mn}^*\}$, and write:

$$\begin{aligned} & 1 + t_{mn}^{(1)} \eta_{mn} \eta_{m+1n} \\ &= \int da_{mn}^* da_{mn} e^{a_{mn} a_{mn}^*} (1 + t_{mn}^{(1)} a_{mn} \eta_{mn}) (1 + a_{mn}^* \eta_{m+1n}) \\ &= Sp_{(a_{mn})} \{ A_{mn} A_{m+1n}^* \}, \end{aligned} \tag{4.1a}$$

$$\begin{aligned} & 1 + t_{mn}^{(2)} \eta_{mn} \eta_{mn+1} \\ &= \int db_{mn}^* db_{mn} e^{b_{mn} b_{mn}^*} (1 + t_{mn}^{(2)} b_{mn} \eta_{mn}) (1 + b_{mn}^* \eta_{mn+1}) \\ &= Sp_{(b_{mn})} \{ B_{mn} B_{mn+1}^* \}. \end{aligned} \tag{4.1b}$$

In the last lines we introduce the abbreviated notation for the arising factors (to be called shortly Grassmann factors), $A_{mn}, B_{mn}, A_{m+1n}^*, B_{mn+1}^*$, while $Sp(\dots)$ stands for the symbol of the Gaussian fermionic averaging. Neglecting the averaging symbol, the Boltzmann weights from (3.2) are now presented in a factorized form: $A_{mn} A_{m+1n}^*, B_{mn} B_{mn+1}^*$. In general, for the whole lattice, there are four Grassmann factors, $A_{mn}, A_{mn}^*, B_{mn}, B_{mn}^*$, which all involve the same variable η_{mn} associated with a given mn site. The idea of the next step is to place nearby the above four factors and to average over η_{mn} in each group independently thus passing to a purely fermionic expression for Q . The obstacle to this method is that the individual Grassmann factors are neither commuting nor anticommuting with each other. It might,

therefore, be difficult, in general, to find the four relevant factors nearby in a global product. The problem of a suitable ordering of the non-commuting Grassmann factors thus arises. This ordering problem can be solved applying the mirror-ordered factorized procedure [7]. The related ideas were first developed in the context of the 2D Ising model [11,12]. The result is that the density matrix from (3.2) can be represented in the following ‘mirror-ordered’ factorized form [7,8]:

$$Q(\eta) = \underset{(a,b)}{Sp} \left\{ \prod_{n=1}^N \left[\prod_{m=1}^M \overleftarrow{B}_{mn}^* \cdot \prod_{m=1}^M \overrightarrow{A}_{mn}^* B_{mn} A_{mn} \right] \right\}. \quad (4.2)$$

The η -averaging can be performed at the junction of the m -ordered products in (4.2) [7]. This results in a purely fermionic expression for the partition function, which can in turn be simplified by integrating out the extra fermionic variables. In this way we obtain [7]:

$$Q = \int \prod_{n=1}^N \prod_{m=1}^M \overleftarrow{dc}_{mn} \exp \left\{ \sum_{m=1}^M \sum_{n=1}^N \left[t_{mn}^{(1)} c_{mn} c_{m+1n} + (-1)^{m+1} t_{mn}^{(2)} c_{mn} c_{mn+1} \right] \right\}, \quad (4.3)$$

with $c_{M+1n} = c_{mN+1} = 0$, where c_{mn} are new totally anticommuting Grassmann variables. The product in the measure is ordered as follows. First, we multiply dc_{mn} over m , with fixed n , then we multiply the resulting products over n . In the field-theoretical language, the fermionic form in the exponential is called action, since the action is quadratic in fermions, we deal here with a free-fermion field theory on a lattice. It is also possible to eliminate the sign factor $(-1)^{m+1}$ from the action. This can be done by the rescaling of the variables in the integral: $c_{mn} \rightarrow c_{mn} i^{m^2+3/2}$, $dc_{mn} \rightarrow dc_{mn} i^{-m^2-3/2}$. This yields equivalent representation [8]:

$$Q = \int \prod_{n=1}^N \prod_{m=1}^M \overleftarrow{dc}_{mn} \exp \left\{ \sum_{m=1}^M \sum_{n=1}^N \left[t_{mn}^{(1)} c_{mn} c_{m+1n} + i t_{mn}^{(2)} c_{mn} c_{mn+1} \right] \right\}. \quad (4.4)$$

The dimer partition function is now expressed as a simple fermionic Gaussian integral. The exact solution for a regular dimer lattice can now be obtained by passing to the momentum space for fermions (Fourier substitution). This is illustrated in the next section with examples of regular (homogeneous) rectangular and brick-hexagonal dimer lattices. It is interesting that these two models exhibit quite different properties. While there is no phase transition for the standard rectangular dimer lattice, there is an exotic phase transition with frozen low-temperature phase for the hexagonal lattice.

5 Analytic results for regular lattices

Let us consider the 2D dimer model on the standard homogeneous rectangular lattice. Assuming $t_{mn}^{(1)} = t_1$, $t_{mn}^{(2)} = t_2$ in (4.4), the partition function is:

$$Q = \int \prod_{n=1}^N \prod_{m=1}^M \overleftarrow{dc}_{mn}^m \exp \sum_{m=1}^M \sum_{n=1}^N \left[t_1 c_{mn} c_{m+1n} + i t_2 c_{mn} c_{m, n+1} \right], \quad (5.1)$$

where $c_{M+1n} = 0$, $c_{m, N+1} = 0$. We deal here with a finite lattice with a free boundary of size $M \times N$, with MN sites. In what follows we assume that M and N both are even. It is interesting that integral (5.1) can be performed exactly despite of that the translational invariance is broken by free boundary. The diagonalizing Fourier substitution is [7,8]:

$$c_{mn} = \frac{2i^{m+n+\frac{3}{2}}}{\sqrt{(M+1)(N+1)}} \sum_{p=1}^M \sum_{q=1}^N c_{pq} \sin\left(\frac{\pi p m}{M+1}\right) \sin\left(\frac{\pi q n}{N+1}\right). \quad (5.2)$$

In the momentum space, we find [8]:

$$Q = \int \prod_{q=1}^N \prod_{p=1}^M \overrightarrow{dc}_{pq}^p \exp \sum_{p=1}^M \sum_{q=1}^N \left[c_{\bar{p}\bar{q}} c_{pq} \left(t_1 \cos \frac{\pi p}{M+1} + i t_2 \cos \frac{\pi q}{N+1} \right) \right], \quad (5.3)$$

where $c_{\bar{p}\bar{q}}$, c_{pq} are the new variables of integration, $\bar{p} = M+1-p$, $\bar{q} = N+1-q$ are ‘conjugated’ momenta selected by the orthogonality relations for the Fourier eigenfunctions from (5.2). Notice that the variables $c_{\bar{p}\bar{q}}$, c_{pq} enter, simultaneously, into the pq and $\bar{p}\bar{q}$ terms of the total sum in (5.3) (in the given case, these terms appear to be equal to each other). To single out explicitly the true independent variables (which is needed to perform the integral) we have to combine these pq and $\bar{p}\bar{q}$ terms together and then to reduce the sum to a half-interval with respect to M or N (choice between M or N is here arbitrary) in order to avoid the double counting of the same terms. Making the reduction with respect to M , the partition function becomes:

$$Q = \int \prod_{p=1}^{\frac{1}{2}M} \prod_{q=1}^N dc_{pq} d\overleftarrow{c}_{\bar{p}\bar{q}} \exp \sum_{p=1}^{\frac{1}{2}M} \sum_{q=1}^N \left[c_{\bar{p}\bar{q}} c_{pq} \left(2t_1 \cos \frac{\pi p}{M+1} + 2it_2 \cos \frac{\pi q}{N+1} \right) \right]. \quad (5.4)$$

The value of the integral readily follows from elementary rules like (2.2). In the resulting product we then make the $q \leftrightarrow \bar{q}$ (or $i \leftrightarrow -i$) symmetrization, and find:

$$Q = \prod_{p=1}^{\frac{1}{2}M} \prod_{q=1}^{\frac{1}{2}N} \left[4t_1^2 \cos^2 \frac{\pi p}{M+1} + 4t_2^2 \cos^2 \frac{\pi q}{N+1} \right], \quad \text{even } M, N. \quad (5.5)$$

This is the exact solution for the 2D dimer model on a finite rectangular lattice with free boundary. This result was first obtained by the combinatorial method [2,3].

Taking the limit of infinite lattice, we obtain the free energy per site for the standard rectangular dimer lattice:

$$-\beta f_{\text{REC}} = \frac{1}{MN} \ln Q \Big|_{MN \rightarrow \infty} = \frac{1}{\pi^2} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} dp dq \ln \left[4 t_1^2 \cos^2 p + 4 t_2^2 \cos^2 q \right] . \quad (5.6)$$

The free energy (5.6) can be represented in other equivalent forms, in particular, it can be written in the form [8]:

$$-\beta f_{\text{REC}} = \frac{1}{4} \ln(t_1 t_2) + \frac{G}{\pi} + \frac{1}{2\pi} \int_{t_2}^{t_1} \int_{t_2}^{t_1} \frac{dx dy}{x^2 + y^2} , \quad (5.7)$$

where $G = 0.915\,965\,594$ is Catalan's constant. The term G/π corresponds to a purely combinatorial contribution to the free energy (and entropy) while the thermodynamics is governed by the doubled-integral term. In thermodynamic interpretation, the dimer weights are to be chosen in the form $t_\alpha = \exp(\varepsilon_\alpha)$, $\varepsilon_\alpha = \beta E_\alpha$, $\alpha = 1, 2$, where E_α are the dimer energies (more precisely, the true energies are $-E_\alpha$), and $\beta = 1/kT$, where kT is the temperature in energy units. A nontrivial thermodynamic interpretation is only possible for the nonequal dimer energies (weights), $E_1 \neq E_2$ ($t_1 \neq t_2$). If $t_1 = t_2$, the energy of any closed-packed dimer configuration is the same and there is no thermodynamics. The thermodynamic functions can all be deduced from the free energy. There is no phase transition in the dimer model on a rectangular lattice [2,3,6,8]. In particular, the specific heat is a smooth function of temperature getting its maximum at $(kT/\Delta E)_{\text{max}} = 0.477572$, at which point $(C/k)_{\text{max}} = 0.169375$ [8], where $\Delta E = |E_1 - E_2|$, C/k is the dimensionless specific heat, k is the Boltzmann constant.

Let us now consider the closed-packed dimer model on the standard hexagonal lattice. There are three different weights, t_1, t_2, t_3 , connected to each site. The hexagonal lattice is equivalent to the so-called brick lattice which can be obtained from a rectangular lattice net by the elimination of some of the vertical bonds [6,7]. Therefore, the fermionic integral for Q for the hexagonal lattice can be deduced from (4.3) and/or (4.4). Another possibility, which we follow below, is to interpret the hexagonal lattice (yet on a rectangular net) in a diagonally layered fashion, as it was done for the 2D Ising model in [12]. Within such interpretation, omitting further details, we obtain the fermionic integral for the partition function of the 2D

hexagonal dimer model in a particularly simple form [8]:

$$Q = \int \prod_{mn} dc_{mn}^* dc_{mn} \exp \sum_{mn} \left[c_{mn} \left(t_1 c_{mn}^* - t_2 c_{m-1n}^* - t_3 c_{mn-1}^* \right) \right], \quad (5.8)$$

where c_{mn}, c_{mn}^* are anticommuting Grassmann variables. Assuming the periodic closing conditions for fermions, which in the given case is the boundary approximation, we then pass to the momentum space by the simplest Fourier substitution:

$$c_{mn} = \frac{1}{\sqrt{L^2}} \sum_{pq} c_{pq} e^{i \frac{2\pi}{L}(mp+nq)}, \quad c_{mn}^* = \frac{1}{\sqrt{L^2}} \sum_{pq} c_{pq}^* e^{-i \frac{2\pi}{L}(mp+nq)}. \quad (5.9)$$

The integral (5.8) becomes:

$$Q = \int \prod_{pq} dc_{pq}^* dc_{pq} \exp \sum_{pq} \left[c_{pq} c_{pq}^* \left(t_1 - t_2 e^{i \frac{2\pi}{L}p} - t_3 e^{i \frac{2\pi}{L}q} \right) \right], \quad (5.10)$$

with evident explicit solution for Q . Respectively, the free energy per site for the hexagonal lattice appears in the form:

$$-\beta f_{\text{HEX}} = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \frac{dp dq}{(2\pi)^2} \ln \left[t_1 - t_2 e^{ip} - t_3 e^{iq} \right]. \quad (5.11)$$

In the thermodynamic interpretation, the weights are: $t_\alpha = \exp(\varepsilon_\alpha)$, where $\varepsilon_\alpha = \beta E_\alpha$ are the dimensionless energies, the true energies of dimers are $-E_\alpha$, and $\beta = 1/kT$. A nontrivial thermodynamics is only possible if the dimer energies (weights) are not all equal to each other. From (5.11) it follows that there is the phase transition, with frozen low-temperature phase, in the dimer model on the hexagonal lattice. To be definite, let $t_1 \leq t_2 < t_3$ (or $-E_3 < -E_2 \leq -E_1$), the critical temperature is then given by the condition $t_1 + t_2 - t_3 = 0$. At low temperatures, $t_1 + t_2 < t_3$, the system is frozen. In this phase ($T < T_c$) all dimers are captured at the bonds with the highest weight (the lowest energy) and the specific heat identically equals zero. In the high-temperature phase, $t_1 + t_2 > t_3$, the system is unfrozen. In this phase ($T > T_c$) the specific heat exhibits the $|T - T_c|^{-\frac{1}{2}}$ singularity as T approaches T_c from above. In the $T > T_c$ phase the specific heat is [8]:

$$\left. \frac{C}{k} \right|_{\text{HEX}} = \frac{1}{\pi} \frac{A(\varepsilon_1, \varepsilon_2, \varepsilon_3)}{\sqrt{(t_1 + t_2 + t_3)(-t_1 + t_2 + t_3)(t_1 - t_2 + t_3)(t_1 + t_2 - t_3)}}, \quad (5.12)$$

where the numerator is a nonsingular function of temperature: $A(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)t_1^2 + (\varepsilon_2 - \varepsilon_1)(\varepsilon_2 - \varepsilon_3)t_2^2 + (\varepsilon_3 - \varepsilon_1)(\varepsilon_3 - \varepsilon_2)t_3^2$. The unusual phase transition in the hexagonal dimer model with frozen $T < T_c$ phase was first mentioned by Kasteleyn in 1963, for a recent comprehensive discussion see [6]. The anisotropic specific heat in the form (5.12) is cited here from [8].

6 Conclusions

We have discussed some aspects of the fermionic interpretation of the 2D closed-packed dimer problem. In its original state, the 2D dimer model is rather a discrete combinatorial problem. Introducing the fermionic variables into the original partition function, we add, effectively, some amount of new degrees of freedom to the original discrete sum. In a sense, this extension of the inherent symmetry just makes the dimer problem tractable analytically. The dimer partition function can be transformed into a fermionic Gaussian integral even for the inhomogeneous distribution of dimer activities over the lattice bonds. For simple homogeneous lattices, the analytic solutions then follow by passing to the momentum space for fermions.

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